

E.M. Ovsyuk\*

# On finding parameters of Mueller matrices of the Lorentzian type from results of polarization measurements

Mozyr State Pedagogical University named after I.P. Shamyakin, Belarus

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## Abstract

With assumption that an optical element is described by a Mueller matrix of the Lorentzian type, a method to find a 3-dimensional complex vector-parameter for a corresponding Mueller matrix from results of four specially chosen polarization measurements has been elaborated.

It is known that the four Stokes parameters describe the state of polarization of the light. They were introduced by Stokes in 1852 [1]. Mueller calculus is a matrix method for dealing with Stokes 4-vectors, it was developed in 1943 by H. Mueller [2]. Any optical element can be represented by a Mueller matrix. In optics, polarized light can be described using the Jones calculus, invented by Jones in 1941 [3]–[6]. Polarized light is represented by a 2-dimensional Jones complex vector, and linear optical elements are represented by  $2 \times 2$  Jones matrices. The Jones calculus is only applicable to light that is completely polarized. Commonly, light which is partially polarized is treated only with the use of vector Mueller calculus.

It is well known that when describing (completely or partially) polarized light a noticeable role may be given to the group of  $(3+1)$ -pseudo orthogonal transformations consisting of a group  $SO(3,1)$  (isomorphic to the Lorentz group). Therefore, techniques developed in the frames of the Lorentz group

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\*e.ovsyuk@mail.ru

(for instance, see [7]–[12]), in particular within relativistic kinematics, may play heuristic role in exploring optical problems (the bibliography on the subject is enormous, many references are given in [13]–[16]).

In a previous paper [17], a general group-theoretic method for recovery of the Mueller matrices of the Lorentzian type for any (Lorentzian type) optical element from results of several independent polarization experiments was investigated. A main feature of treatment given in [17] is that initial (probing) beams of light are arbitrary. Meanwhile, in the book by Snopko [18], an entirely different way to restore a 16-element Mueller matrices (not necessarily of the Lorentzian type) is described. This method is based on the use of specially chosen probing beams of light: one natural and three completely polarized.

A natural question about the correlation of these two techniques<sup>1</sup> arises. The main goal of the present paper is to investigate restriction of the general method [18] to the special case of Mueller matrices of the Lorentzian type.

Let us describe the general rules for finding Mueller matrices of an arbitrary optical element (first without any restriction to the class of Mueller matrices of the Lorentzian type) [18]. The first probing light beam is chosen as the natural light

$$S_{(0)}^a = (I, 0, 0, 0) \quad \Longrightarrow \quad S_{(0)}^{a'} ,$$

$$M = \begin{vmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{vmatrix} \begin{vmatrix} I = S_{(0)}^0 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} S_{(0)}^{0'} \\ S_{(0)}^{1'} \\ S_{(0)}^{2'} \\ S_{(0)}^{3'} \end{vmatrix} ,$$

$$m_{00}I = S_{(0)}^{0'} , \quad m_{10}I = S_{(0)}^{1'} , \quad m_{20}I = S_{(0)}^{2'} , \quad m_{30}I = S_{(0)}^{3'} . \quad (1)$$

The next three probing beams are chosen as completely polarized ones, and of a special form:

$$S_{(1)}^a = (I, I, 0, 0) \quad \Longrightarrow \quad S_{(1)}^{a'} ,$$

$$M = \begin{vmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{vmatrix} \begin{vmatrix} I \\ I \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} S_{(1)}^{0'} \\ S_{(1)}^{1'} \\ S_{(1)}^{2'} \\ S_{(1)}^{3'} \end{vmatrix} ,$$

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<sup>1</sup>The author is grateful to E.A. Tolkachev and Y.A. Kurochkin for pointing out on this fact.

$$\begin{aligned}
S_{(0)}^{0'} + m_{01}I &= S_{(1)}^{0'} , & S_{(0)}^{1'} + m_{11}I &= S_{(1)}^{1'} , \\
S_{(0)}^{2'} + m_{21}I &= S_{(1)}^{2'} , & S_{(0)}^{3'} + m_{31}I &= S_{(1)}^{3'} ;
\end{aligned} \tag{2}$$

$$\begin{aligned}
S_{(2)}^a &= (I, 0, I, 0) \quad \Longrightarrow \quad S_{(2)}^{a'} , \\
M &= \begin{vmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{vmatrix} \begin{vmatrix} I \\ 0 \\ I \\ 0 \end{vmatrix} = \begin{vmatrix} S_{(2)}^{0'} \\ S_{(2)}^{1'} \\ S_{(2)}^{2'} \\ S_{(2)}^{3'} \end{vmatrix} , \\
S_{(0)}^{0'} + m_{02}I &= S_{(2)}^{0'} , & S_{(0)}^{1'} + m_{12}I &= S_{(2)}^{1'} , \\
S_{(0)}^{2'} + m_{22}I &= S_{(2)}^{2'} , & S_{(0)}^{3'} + m_{32}I &= S_{(2)}^{3'} ;
\end{aligned} \tag{3}$$

$$\begin{aligned}
S_{(3)}^a &= (I, 0, 0, I) \quad \Longrightarrow \quad S_{(3)}^{a'} , \\
M &= \begin{vmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{vmatrix} \begin{vmatrix} I \\ 0 \\ 0 \\ I \end{vmatrix} = \begin{vmatrix} S_{(3)}^{0'} \\ S_{(3)}^{1'} \\ S_{(3)}^{2'} \\ S_{(3)}^{3'} \end{vmatrix} , \\
S_{(0)}^{0'} + m_{03}I &= S_{(3)}^{0'} , & S_{(0)}^{1'} + m_{13}I &= S_{(3)}^{1'} , \\
S_{(0)}^{2'} + m_{23}I &= S_{(3)}^{2'} , & S_{(0)}^{3'} + m_{33}I &= S_{(3)}^{3'} .
\end{aligned} \tag{4}$$

The resulting system of equations leads to the following explicit expressions for the 16 elements of the Mueller matrix

$$\begin{aligned}
m_{00} &= S_{(0)}^{0'}/I , \quad m_{10} = S_{(0)}^{1'}/I , \quad m_{20} = S_{(0)}^{2'}/I , \quad m_{30} = S_{(0)}^{3'}/I , \\
m_{01} &= \frac{S_{(1)}^{0'} - S_{(0)}^{0'}}{I} , \quad m_{11} = \frac{S_{(1)}^{1'} - S_{(0)}^{1'}}{I} , \quad m_{21} = \frac{S_{(1)}^{2'} - S_{(0)}^{2'}}{I} , \quad m_{31} = \frac{S_{(1)}^{3'} - S_{(0)}^{3'}}{I} , \\
m_{02} &= \frac{S_{(2)}^{0'} - S_{(0)}^{0'}}{I} , \quad m_{12} = \frac{S_{(2)}^{1'} - S_{(0)}^{1'}}{I} , \quad m_{22} = \frac{S_{(2)}^{2'} - S_{(0)}^{2'}}{I} , \quad m_{32} = \frac{S_{(2)}^{3'} - S_{(0)}^{3'}}{I} , \\
m_{03} &= \frac{S_{(3)}^{0'} - S_{(0)}^{0'}}{I} , \quad m_{13} = \frac{S_{(3)}^{1'} - S_{(0)}^{1'}}{I} , \quad m_{23} = \frac{S_{(3)}^{2'} - S_{(0)}^{2'}}{I} , \quad m_{33} = \frac{S_{(3)}^{3'} - S_{(0)}^{3'}}{I} .
\end{aligned} \tag{5}$$

We consider the case when the matrix is isomorphic to the Mueller matrix of the Lorentz group. This means that we must assume that all Stokes vectors are invariant in a sense of "relativistic length":  $g_{ab}S^aS^b = g_{ab}S^{a'}S^{b'}$ , so we get

$$\begin{aligned} I^2 &= (S_{(0)}^{0'})^2 - (\mathbf{S}'_{(0)})^2, & 0 &= (S_{(1)}^{0'})^2 - (\mathbf{S}'_{(1)})^2, \\ 0 &= (S_{(2)}^{0'})^2 - (\mathbf{S}'_{(2)})^2, & 0 &= (S_{(3)}^{0'})^2 - (\mathbf{S}'_{(3)})^2. \end{aligned} \quad (6)$$

We first consider a simpler problem, assuming that the matrix is isomorphic to the element of a group of 3-dimensional rotations:

$$M = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & m_{11} & m_{12} & m_{13} \\ 0 & m_{21} & m_{22} & m_{23} \\ 0 & m_{31} & m_{32} & m_{33} \end{vmatrix}. \quad (7)$$

The system (5) takes the form

$$\begin{aligned} 1 &= 1, & 0 &= 0, & 0 &= 0, & 0 &= 0, \\ 0 &= 0, & m_{11} &= \frac{S_{(1)}^{1'}}{I}, & m_{21} &= \frac{S_{(1)}^{2'}}{I}, & m_{31} &= \frac{S_{(1)}^{3'}}{I}, \\ 0 &= 0, & m_{12} &= \frac{S_{(2)}^{1'}}{I}, & m_{22} &= \frac{S_{(2)}^{2'}}{I}, & m_{32} &= \frac{S_{(2)}^{3'}}{I}, \\ 0 &= 0, & m_{13} &= \frac{S_{(3)}^{1'}}{I}, & m_{23} &= \frac{S_{(3)}^{2'}}{I}, & m_{33} &= \frac{S_{(3)}^{3'}}{I}. \end{aligned} \quad (8)$$

Additional conditions (6) are simplified

$$I^2 = I^2, \quad I^2 = (\mathbf{S}'_{(1)})^2, \quad I^2 = (\mathbf{S}'_{(2)})^2, \quad I^2 = (\mathbf{S}'_{(3)})^2. \quad (9)$$

According to (8), the matrix (7) can be represented in the form (we follow only the three-dimensional matrix)

$$M = \frac{1}{I} \begin{vmatrix} S_{(1)}^{1'} & S_{(2)}^{1'} & S_{(3)}^{1'} \\ S_{(1)}^{2'} & S_{(2)}^{2'} & S_{(3)}^{2'} \\ S_{(1)}^{3'} & S_{(2)}^{3'} & S_{(3)}^{3'} \end{vmatrix}. \quad (10)$$

The determinant of this matrix (belonging to rotation group) must be equal to 1

$$\det M = \frac{1}{I} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{1}{I^3} \mathbf{a}(\mathbf{b} \times \mathbf{c}) = 1. \quad (11a)$$

If we use the polarization vector  $\mathbf{S} = I\mathbf{p}$ , then the resulting constraint can be expressed as

$$\mathbf{p}'_{(1)}(\mathbf{p}'_{(2)} \times \mathbf{p}'_{(3)}) = 1 . \quad (11b)$$

These three vectors cannot be considered as independent quantities since they are obtained as a result of rotating of three initial polarization vectors:

$$\mathbf{p}_{(1)} = (1, 0, 0) , \quad \mathbf{p}_{(2)} = (0, 1, 0) , \quad \mathbf{p}_{(3)} = (0, 0, 1) . \quad (11c)$$

Three vectors  $\mathbf{p}'_{(1)}, \mathbf{p}'_{(2)}, \mathbf{p}'_{(3)}$  must have magnitude 1, orthogonal to each other, and be right-handed ones.

Matrix (10) must be identified with the orthogonal matrix of the group  $SO(3, R)$  (see in [10])

$$O = \begin{vmatrix} 1 - 2(n_2^2 + n_3^2) & -2n_0n_3 + 2n_1n_2 & +2n_0n_2 + 2n_1n_3 \\ +2n_0n_3 + 2n_1n_2 & 1 - 2(n_3^2 + n_1^2) & -2n_0n_1 + 2n_2n_3 \\ -2n_0n_2 + 2n_1n_3 & +2n_0n_1 + 2n_2n_3 & 1 - 2(n_1^2 + n_2^2) \end{vmatrix} , \quad (12a)$$

parameters satisfy the condition

$$n_0^2 + n_1^2 + n_2^2 + n_3^2 = +1 . \quad (12b)$$

Further, we use (with minor modifications) technique developed in [10]. We compute  $\text{Sp } M$  and find  $n_0$ :

$$\begin{aligned} \text{Sp } M &= (S_{(1)}^{1'} + S_{(2)}^{2'} + S_{(3)}^{3'})/I = p_{(1)}^{1'} + p_{(2)}^{2'} + p_{(3)}^{3'} , \\ 2n_0 &= \sqrt{p_{(1)}^{1'} + p_{(2)}^{2'} + p_{(3)}^{3'}} + 1 . \end{aligned} \quad (13)$$

Let us separate an antisymmetric part of the matrix  $M_{as} = (M - \tilde{M})/2$  and equate it to the  $O_{as} = (O - \tilde{O})/2$

$$\begin{aligned} & \frac{1}{2} \begin{vmatrix} 0 & -(p_{(1)}^{2'} - p_{(2)}^{1'}) & (p_{(3)}^{1'} - p_{(1)}^{3'}) \\ (p_{(1)}^{2'} - p_{(2)}^{1'}) & 0 & -(p_{(2)}^{3'} - p_{(3)}^{2'}) \\ -(p_{(3)}^{1'} - p_{(1)}^{3'}) & (p_{(2)}^{3'} - p_{(3)}^{2'}) & 0 \end{vmatrix} \\ &= \begin{vmatrix} 0 & -2n_0n_3 & +2n_0n_2 \\ +2n_0n_3 & 0 & -2n_0n_1 \\ -2n_0n_2 & +2n_0n_1 & 0 \end{vmatrix} . \end{aligned}$$

As a result, we obtain

$$\begin{aligned}
2n_0n_1 &= \frac{1}{2}(p_{(2)}^{3'} - p_{(3)}^{2'}) \implies n_1 = \frac{p_{(2)}^{3'} - p_{(3)}^{2'}}{2\sqrt{p_{(1)}^{1'} + p_{(2)}^{2'} + p_{(3)}^{3'} + 1}}, \\
2n_0n_2 &= \frac{1}{2}(p_{(3)}^{1'} - p_{(1)}^{3'}) \implies n_2 = \frac{p_{(3)}^{1'} - p_{(1)}^{3'}}{2\sqrt{p_{(1)}^{1'} + p_{(2)}^{2'} + p_{(3)}^{3'} + 1}}, \\
2n_0n_3 &= \frac{1}{2}(p_{(1)}^{2'} - p_{(2)}^{1'}) \implies n_3 = \frac{p_{(1)}^{2'} - p_{(2)}^{1'}}{2\sqrt{p_{(1)}^{1'} + p_{(2)}^{2'} + p_{(3)}^{3'} + 1}}.
\end{aligned} \tag{14}$$

It is easily verified the equality (12b)

$$\begin{aligned}
&[p_{(1)}^{1'} + p_{(2)}^{2'} + p_{(3)}^{3'} + 1] + \frac{(p_{(2)}^{3'} - p_{(3)}^{2'})^2}{p_{(1)}^{1'} + p_{(2)}^{2'} + p_{(3)}^{3'} + 1} \\
&+ \frac{(p_{(3)}^{1'} - p_{(1)}^{3'})^2}{p_{(1)}^{1'} + p_{(2)}^{2'} + p_{(3)}^{3'} + 1} + \frac{(p_{(1)}^{2'} - p_{(2)}^{1'})^2}{p_{(1)}^{1'} + p_{(2)}^{2'} + p_{(3)}^{3'} + 1} = 4.
\end{aligned} \tag{15}$$

Note that, according to the procedure [17], to restore the Mueller matrices in 3-dimensional case, it suffices to use only two pairs of vectors. Here we employ three pairs.

Let us go back to the 4-dimensional Mueller matrices

$$M = \frac{1}{I} \begin{vmatrix} S_{(0)}^{0'} & S_{(1)}^{0'} - S_{(0)}^{0'} & S_{(2)}^{0'} - S_{(0)}^{0'} & S_{(3)}^{0'} - S_{(0)}^{0'} \\ S_{(0)}^{1'} & S_{(1)}^{1'} - S_{(0)}^{1'} & S_{(2)}^{1'} - S_{(0)}^{1'} & S_{(3)}^{1'} - S_{(0)}^{1'} \\ S_{(0)}^{2'} & S_{(1)}^{2'} - S_{(0)}^{2'} & S_{(2)}^{2'} - S_{(0)}^{2'} & S_{(3)}^{2'} - S_{(0)}^{2'} \\ S_{(0)}^{3'} & S_{(1)}^{3'} - S_{(0)}^{3'} & S_{(2)}^{3'} - S_{(0)}^{3'} & S_{(3)}^{3'} - S_{(0)}^{3'} \end{vmatrix} \tag{16}$$

and introduce the notation

$$S_{(0)}^{a'} = F^a, \quad S_{(1)}^{a'} = A^a, \quad S_{(2)}^{a'} = B^a, \quad S_{(3)}^{a'} = C^a, \tag{17}$$

then

$$M = \frac{1}{I} \begin{vmatrix} F^0 & A^0 - F^0 & B^0 - F^0 & C^0 - F^0 \\ F^1 & A^1 - F^1 & B^1 - F^1 & C^1 - F^1 \\ F^2 & A^2 - F^2 & B^2 - F^2 & C^2 - F^2 \\ F^3 & A^3 - F^3 & B^3 - F^3 & C^3 - F^3 \end{vmatrix}. \tag{18}$$

We identify this matrix with the matrix of the Lorentz group and use a method of finding parameters of the matrix Lorentz described in [19][20]. In the main points, it coincides with well-developed technique given in [10], the differences are related with the transition to spinor covering  $SL(2.C)$  for the Lorentz group  $L_+^\uparrow$ . Let us briefly describe this recipe. Any orthochronous Lorentz transformation can be represented as follows:

$$L_b^a(k, k^*) = \bar{\delta}_b^c ( -\delta_c^a k^n k_n^* + k_c k^{a*} + k_c^* k^a + i \epsilon_c^{anm} k_n k_m^* ) , \quad (19)$$

where  $\bar{\delta}_b^c$  – special (different from the usual) the Kronecker delta-symbol

$$\bar{\delta}_b^c = \begin{cases} 0, & c \neq b ; \\ +1, & c = b = 0 ; \\ -1, & c = b = 1, 2, 3 . \end{cases}$$

We expand the four-dimensional parameter  $k_a$  into real and imaginary parts:

$$k_0 = m_0 - i n_0 = \Delta e^{i\kappa} , \quad \mathbf{k} = (k_j) = \mathbf{m} - i \mathbf{n} \quad (20)$$

and represent the matrix  $\Lambda$  ( $L_a^b = \bar{\delta}_a^c \Lambda_c^b$ ) as the sum of symmetric and antisymmetric parts of the  $\Lambda = (S + A)$ :

$$S = \begin{vmatrix} \Delta^2 + \mathbf{m}^2 + \mathbf{n}^2 & 2 [\mathbf{n} \mathbf{m}] \\ 2 [\mathbf{n} \mathbf{m}] & -\Delta^2 + \mathbf{m}^2 + \mathbf{n}^2 - 2 \mathbf{m} \bullet \mathbf{m} - 2 \mathbf{n} \bullet \mathbf{n} \end{vmatrix} ,$$

$$A = 2 \Delta \begin{vmatrix} 0 & -(\mathbf{m} \cos \kappa - \mathbf{n} \sin \kappa) \\ (\mathbf{m} \cos \kappa - \mathbf{n} \sin \kappa) & (\mathbf{m} \cos \kappa + \mathbf{n} \sin \kappa)^\times \end{vmatrix} . \quad (21)$$

Notation is used:  $(\mathbf{n} \bullet \mathbf{n})_{ij} = n_i n_j$  ,  $(\mathbf{m} \bullet \mathbf{m})_{ij} = m_i m_j$  ,  $(\mathbf{b}^\times)_{ij} = \epsilon_{ijk} b_k$  .

Taking into account the dependence of matrix elements of  $A$  on parameter  $\kappa$ , the phase of a complex number  $k_0$ , we introduce three-dimensional vectors  $\mathbf{M}$  and  $\mathbf{N}$

$$\begin{vmatrix} \mathbf{M} \\ \mathbf{N} \end{vmatrix} = \begin{vmatrix} \cos \kappa & -\sin \kappa \\ \sin \kappa & \cos \kappa \end{vmatrix} \begin{vmatrix} \mathbf{m} \\ \mathbf{n} \end{vmatrix} , \quad (22)$$

then for the matrix  $S$  and  $A$  we obtain the representations

$$S = \begin{vmatrix} \Delta^2 + \mathbf{M}^2 + \mathbf{N}^2 & 2 [\mathbf{N} \mathbf{M}] \\ 2 [\mathbf{N} \mathbf{M}] & -\Delta^2 + \mathbf{M}^2 + \mathbf{N}^2 - 2 \mathbf{M} \bullet \mathbf{M} - 2 \mathbf{N} \bullet \mathbf{N} \end{vmatrix} ,$$

$$A = 2 \Delta \begin{vmatrix} 0 & -\mathbf{M} \\ +\mathbf{M} & \mathbf{N}^\times \end{vmatrix} . \quad (23)$$

Relations (22) can be written in the form of a complex equation:

$$e^{-i\kappa} \mathbf{k} = e^{-i\kappa} (\mathbf{m} - i \mathbf{n}) = \mathbf{M} - i \mathbf{N} ,$$

respectively, the constraint on the determinant of the Lorentz matrix can be written as

$$\Delta^2 - (\mathbf{M} - i \mathbf{N})^2 = e^{-2i\kappa} . \quad (24)$$

Now, we formulate the rule for finding the explicit form of parameter  $k_a$ .

1) First, since the equality holds

$$\text{Sp } L = 2 (g^{nm} + \bar{g}^{nm}) k_n k_m^* = 4 k_0 k_0^* = 4 \Delta^2 , \quad (25)$$

we must compute  $\text{Sp } L$  and then to find the value of  $\Delta$ . 2) After that, by an antisymmetric part  $A$  of the matrix  $\Lambda$  we define the vectors  $\mathbf{M}$  and  $\mathbf{N}$ .

3) Finally, the found values so  $(\Delta, \mathbf{M}, \mathbf{N})$  restore the parameter  $k_a$

$$(k_0, k_j) = \frac{\pm 1}{\sqrt{\Delta^2 - (\mathbf{M} - i \mathbf{N})^2}} (\Delta, \mathbf{M} - i \mathbf{N}) , \quad (26)$$

where  $(\pm)$  represent the possibility of finding a spinor transformation from the vector one only up to the sign  $\pm$ .

No we will apply the formula (26) to find the parameters of the matrix  $M$  (18). Compute the trace of the matrix  $M$  and the parameter  $\Delta$ :

$$2\Delta = \pm \sqrt{\frac{F^0 + (A^1 - F^1) + (B^2 - F^2) + (C^3 - F^3)}{I}} . \quad (27)$$

From the matrix  $M$  we obtain a matrix  $\Lambda$

$$\Lambda = \frac{1}{I} \begin{vmatrix} F^0 & (A^0 - F^0) & (B^0 - F^0) & (C^0 - F^0) \\ -F^1 & -(A^1 - F^1) & -(B^1 - F^1) & -(C^1 - F^1) \\ -F^2 & -(A^2 - F^2) & -(B^2 - F^2) & -(C^2 - F^2) \\ -F^3 & -(A^3 - F^3) & -(B^3 - F^3) & -(C^3 - F^3) \end{vmatrix} .$$

Find the antisymmetric part of the matrix  $\Lambda$  and identify it with

$$2 \Delta \begin{vmatrix} 0 & -M_1 & -M_2 & -M_3 \\ M_1 & 0 & N_3 & -N_2 \\ M_2 & -N_3 & 0 & N_1 \\ M_3 & N_2 & -N_1 & 0 \end{vmatrix} .$$

So we arrive at

$$\frac{F^0 - F^1 - A^0}{2I} = 2\Delta M_1 ,$$



$$\begin{aligned}
\frac{F^0 - F^2 - B^0}{2I} &= 2\Delta M_2 , \\
\frac{F^0 - F^2 - C^0}{2I} &= 2\Delta M_3 , \\
\frac{F^2 - F^3 - C^2 + B^3}{2I} &= 2\Delta N^1 , \\
\frac{F^3 - F^1 - A^3 + C^1}{2I} &= 2\Delta N^2 , \\
\frac{F^1 - F^2 - B^1 + A^2}{2I} &= 2\Delta N^3 .
\end{aligned} \tag{28}$$

The answer can be presented in a concise form, if you go to a 3-dimensional vector [10] to parameterize the 4-vector Lorentz transformations:

$$i\mathbf{q} = \frac{\mathbf{k}}{k_0} = \frac{\mathbf{M} - i\mathbf{N}}{\Delta}, \tag{29}$$

that is

$$\begin{aligned}
iq_1 &= 2 \frac{(F^0 - A^0) - F^1 - i[(F^2 - F^3) - (C^2 - B^3)]}{F^0 + (A^1 - F^1) + (B^2 - F^2) + (C^3 - F^3)} , \\
iq_2 &= 2 \frac{(F^0 - B^0) - F^2 - i[(F^3 - F^1) - (A^3 - C^1)]}{F^0 + (A^1 - F^1) + (B^2 - F^2) + (C^3 - F^3)} , \\
iq_3 &= 2 \frac{(F^0 - C^0) - F^3 - i[(F^1 - F^2) - (B^1 - A^2)]}{F^0 + (A^1 - F^1) + (B^2 - F^2) + (C^3 - F^3)} .
\end{aligned} \tag{30}$$

To verify these formulas let us consider a simple example. Let a Mueller matrix of the Lorentz-type be

$$M = L = \begin{vmatrix} \cosh \beta & 0 & 0 & \sinh \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \beta & 0 & 0 & \cosh \beta \end{vmatrix},$$

one can compute

$$\begin{aligned}
S_{(0)}^{0'} &= F^0 = I \cosh \beta , \quad S_{(0)}^{1'} = F^1 = 0 , \quad S_{(0)}^{2'} = F^2 = 0 , \quad S_{(0)}^{3'} = F^3 = I \sinh \beta , \\
S_{(1)}^{0'} &= A^0 = I \cosh \beta , \quad S_{(1)}^{1'} = A^1 = I , \quad S_{(1)}^{2'} = A^2 = 0 , \quad S_{(1)}^{3'} = A^3 = I \sinh \beta , \\
S_{(2)}^{0'} &= B^0 = I \cosh \beta , \quad S_{(2)}^{1'} = B^1 = 0 , \quad S_{(2)}^{2'} = B^2 = I , \quad S_{(2)}^{3'} = B^3 = I \sinh \beta , \\
S_{(3)}^{0'} &= C^0 = I (\cosh \beta + \sinh \beta) , \quad S_{(3)}^{1'} = C^1 = 0 ,
\end{aligned}$$

$$S_{(3)}^{2'} = C^2 = 0, \quad S_{(3)}^{3'} = C^3 = I (\cosh \beta + \sinh \beta).$$

Substituting the m into (30) we get an expected result

$$q_1 = 0 \quad q_2 = 0, \quad q_3 = -\frac{\sinh \beta}{\cosh \beta + 1} = i \tanh \frac{\beta}{2}.$$

Let us summarize the main result: with assumption that an optical element is described by a Mueller matrix of the Lorentzian type, a method to find a 3-dimensional complex vector-parameter for a corresponding Mueller matrix from results of four specially chosen polarization measurements has been elaborated.

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